

What Quantity And Measurement Really Are

QUANTITY

What are the marks of quantity? What distinguishes a quantitative from a nonquantitative variable? The marks of quantity are order and additivity: the values of a quantitative variable are ordered and possess additive structure. It is the aim of this chapter to make explicit the meaning of these claims.

There is, however, a preliminary point to clarify. What is a variable? Very generally, it is anything relative to which objects may vary. Length is a variable, different objects may have different lengths. Color is a variable, for different objects may be of different colors. Sex is a variable, some things being female and some male. Consider length in more detail. The variable of length is simply the class of all lengths. Each particular length is a property that some things might possess. So a variable may be a class of properties. Not just *any* class of properties constitutes a variable. The properties of being 5-feet long, of being pink, of being female, and of being highly intelligent taken together do not form a variable because some object may possess more than one of these properties simultaneously. Properties only form a variable if they are mutually exclusive, that is, if nothing can possess more than one of them simultaneously. The class of all lengths is a variable for any object can only be one length at a time.

It is not only classes of mutually exclusive *properties* that form variables. Classes of mutually exclusive *relations* do so as well. The distinction between properties and relations is an important one. Things possess properties singly. The fact that my pen is ten centimeters long is a fact about it and nothing else. On the other hand, relations involve pluralities of things. If the pen is on the desk then this situation involves both the pen and the desk. Binary relations involve

two things, ternary relations three things, and so on. Of course, everything has many properties and stands in many relations to other things.

Another example of a relation is velocity. It is a binary relation. The velocity of A relative to B is a matter involving only A and B . Of course, the velocity of A relative to B may vary and it can only be one value at a time. Hence, it is a variable. It is true that the velocity of A relative to B and the velocity of A relative to C may differ at any point in time, but this does not mean that A has more than one velocity at a time. A , by itself, does not have a velocity. It is pairs of things, A and B , and A and C , and so on, which have velocities.

Another class of relations forming a variable is the following: being the father of, being the mother of, being the sister of, being the brother of, and being the child of. Cannot certain pairs of people stand in more than one of these relations to each other? If A is the father of B , then B is the child of A , and so A and B are involved in both relations at once. However, a binary relation holds not simply between a pair of things but between an *ordered pair*. Hence, a class of binary relations forms a variable only if no more than one of the relations within it ever holds between a particular ordered pair of entities at any one time. A similar point applies to ternary relations and ordered triples, quaternary relations and ordered quadruples, and so on.

Within this book, the particular properties or relations constituting a variable will be referred to as *values of that variable*. For example, being 6-meters long is a value of the length variable; being female is a value of the sex variable; and traveling at 100 meters per second relative to something is a value of the velocity variable. When it is said that a quantitative variable is ordered and additive, it is meant that there are ordinal and additive relations between the values of the variable.

The first fact to note about a quantitative variable is that its values are ordered. For example, lengths are ordered according to their magnitude, six meters is greater than 2 meters, and so on. Similarly, the values of other quantitative variables are ordered according to their magnitudes. The familiar symbols, " $\geq$ " and ">" will be used to denote this relation of magnitude, " $\geq$ " meaning "at least as great as," and ">" meaning "greater than." Also, the symbol " $=$ " will be used to signify identity of value. (For more information on the concept of order the reader is referred to Appendix 1).

Let X , Y , and Z be any three values of a variable, Q . Then Q is *ordinal* if and only if

- (1) if $X \geq Y$ and $Y \geq Z$ then $X \geq Z$ (transitivity);
- (2) if $X \geq Y$ and $Y \geq X$ then $X = Y$ (antisymmetry); and
- (3) either $X \geq Y$ or $Y \geq X$ (strong connexity).

A relation possessing these three properties is called a *simple order*, so Q is ordinal if and only if $\geq$ is a simple order on its values. All quantitative variables

are simply ordered by $\geq$, but not every ordinal variable is quantitative, for quantity involves more than order. It involves additivity.

Additivity is a ternary relation, symbolized as " $X + Y = Z$ ". Let Q be any ordinal variable such that for any of its values X , Y and Z

- (4) $X + (Y + Z) = (X + Y) + Z$ (associativity);
- (5) $X + Y = Y + X$ (commutativity);
- (6) $X \geq Y$ if and only if $X + Z \geq Y + Z$ (monotonicity);
- (7) if $X > Y$ then there exists a value Z such that $X = Y + Z$ (solvability);
- (8) $X + Y > X$ (positivity); and
- (9) there exists a natural number n such that $nX \geq Y$ (where $1X = X$ and $(n + 1)X = nX + X$) (Archimedean condition).

In such a case the ternary relation involved is *additive* and Q is a *quantitative variable*.

These nine conditions (the ordinal and the additive) are *uniformities of coexistence* (J. S. Mill, 1848). What is meant by calling them this is that these conditions do nothing more than describe the structure of the variable. They do not, of themselves, describe the behavior of any objects possessing values of the variable. To presume that they do is to make Campbell's mistake. How the additive structure of a quantitative variable is reflected in the behavior of relevant objects is a matter that depends also on the kinds of objects they are (i.e., upon the laws of nature relating to them in virtue of their other properties, and the other relations into which they enter). The issue of how quantitative structure may be detected is discussed in the next chapter. Here our focus is the concept of quantity itself.

The fact that these conditions are uniformities of coexistence affects the interpretation of the additive relation. $X + Y = Z$ does not mean that X *added to* Y is the same as Z . That is far too operational an interpretation. Rather, $X + Y = Z$ is a relation between X , Y , and Z . It is the relation of Z being entirely composed of discrete parts X and Y .

Hence, condition (4) may be interpreted this way: the value entirely composed of discrete parts X and $(Y + Z)$ (where $Y + Z$ is the value entirely composed of discrete parts Y and Z) is the same as the value entirely composed of discrete parts $(X + Y)$ and Z (where $X + Y$ is the value entirely composed of discrete parts X and Y).

Condition (5) means that the value entirely composed of discrete parts X and Y is the same as the value entirely composed of discrete parts Y and X . So (4) and (5) together imply that the order of the parts constituting any value of a quantitative variable is irrelevant.

In the same vein, (6) means that if $X \geq Y$ then a value entirely composed of discrete parts X and Z is at least as great as a value entirely composed of discrete parts Y and Z .

Condition (7) requires that the variable be additively complete. If $X > Y$ then

X will be entirely composed of discrete parts Y and some other value of Q , Z . (7) requires that Z exist.

Condition (8) states that the value entirely composed of discrete parts X and Y is greater than X itself. This is what it means to say that all values of the variable are *positive*. In this book our interest is only in such variables. Krantz et al. (1971) indicate how quantities with nonpositive values may be described.

Condition (9) ensures that no value, Y , of the variable is infinitely greater than any other value, X . It is called the *Archimedean* condition after the ancient Greek mathematician, Archimedes of Syracuse, who is said to have been the first to state it explicitly. However, it may well have been known earlier than Archimedes for it is implicit in Definition 4 of Book V of Euclid's *Elements*. Within condition (9), the meaning of the expression " nX " may seem obscure. As mentioned, n is a natural number, that is, one of the infinite series 1, 2, 3, ... The expression " nX ", therefore, refers to the value of the variable which is entirely composed of the discrete parts X and X and . . . etc. (until n of them are listed). However, because there is only one value X how can n discrete X s be referred to? To understand this requires understanding the concept of composition in terms of identical parts. This is a familiar enough concept. Consider a liter of water which is composed of a large number of water molecules, all identical to one another.

However, an obscurity remains due to the fact that such molecules are physical objects, while the value, X of some quantitative variable is a property or a relation. It may help then to consider length, the paradigm of quantity. Let the objects involved be a set of straight, rigid rods of various lengths. Then $X + Y = Z$ means that if any rod, a , is of length Z , then a is entirely composed of two parts, c and d , such that c is of length X , and d is of length Y . Of course, here the sense in which a is entirely composed of c and d is quite obvious: there will be a point along the extent of a such that everything from one end up to that point is c and everything beyond it to the other end is d . This is illustrated in Figure 3.1. Thus, in this case, the expression " nX " refers to the length of a rod, b , which is entirely composed of n discrete parts

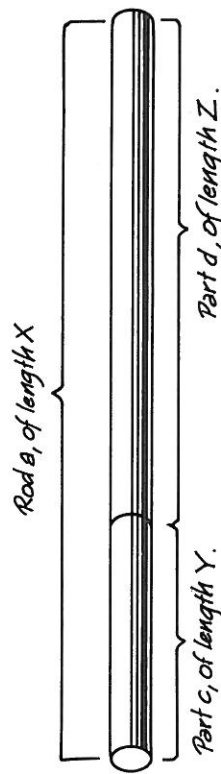


FIG. 3.1 The meaning of the additivity relation, $X = Y + Z$, is illustrated with respect to length by rod a , of length X . Rod a is entirely composed of the discrete parts c and d , where c is of length Y and d is of length Z .

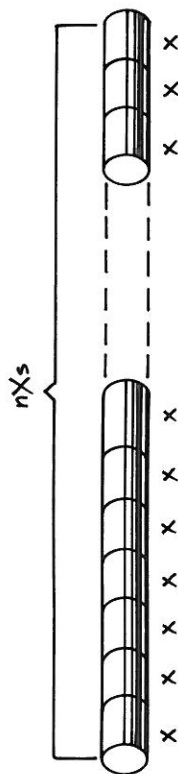


FIG. 3.2 For the case of length, the meaning of the expression " nX " is given by the length of a rod b which is entirely composed of n discrete parts each of length x .

each of length X (cf. Fig. 3.2). So the Archimedean condition asserts that no matter how small length X is and no matter how large length Y is, there is an n such that $nX \geq Y$.

The example of length here may give the false impression that if $X + Y = Z$ and object a possesses value Z , then a can be decomposed into parts c and d , where c possesses X and d , Y . However this is not necessary, for our ability to so decompose object a depends on the kind of thing a is (apart from being Z , that is), and on our technological capacities. Another example will clarify this point. Suppose that X , Y , and Z are densities and a is an iron cube of uniform density. Now if $X + Y = Z$ and a is Z then a will be entirely composed of parts c and d , such that c is X , and d is Y , but these will not be parts into which we can decompose a . The reason, of course, is that a , c , and d must all be of the same volume, though they must differ in mass. Hence, c and d will be "ghosts" of a , identical to it in external size and shape, but differing in inner constitution, in that some of the matter composing a constitutes c (say, every second or third molecule of iron in a) and the rest constitutes d . These parts of a exist, however there is no way that it can be decomposed into them.

It is in this sense that these nine laws of quantity are uniformities of coexistence, rather than realizable operations. All quantities share a similar structure, but not all objects possessing quantitative attributes or entering into quantitative relations manifest that quantitative structure in the same manner.

Within this book, these nine conditions will be taken to be the paradigm of quantity. It is recognized that there are exceptions (e.g., some quantities admit negative values, some are periodic, and some possess maximal values). The interested reader is referred to Krantz et al. (1971) for a definition of such variables. In all these cases some modification of one or more of these nine conditions is necessary. However, these nine conditions are regarded as defining the paradigm of quantity because they describe the common structure of the two basic quantities that, historically, have shaped attempts at quantification. These are aggregate magnitude and length. These conditions describe what is common to these two quantities, and any nonnumerical variable that shares these common features is capable of measurement. In what follows, unless otherwise stated, a quantitative variable will be taken to be one that satisfies these nine conditions.

CONTINUOUS QUANTITY

Aristotle divided quantity in two, discrete and continuous. An example of discrete quantity is aggregate magnitude (ie., what is common to all classes or aggregates of the same size). That which is common to all aggregates of some specific size is the property of being an n membered aggregate, (for some particular natural number, n) and the class of all such properties constitutes a quantity. Any value of this quantity may be assessed by considering its numerical relation to that of the unit value, (the property of being a single membered aggregate) and this is what is discovered by counting. However, the values of a continuous quantity cannot be assessed by counting relative to a common unit. Instead, they must be measured. Measurement is the identification of some value of a continuous quantity relative to an arbitrarily chosen unit, via their numerical relation. In order to understand this statement the relationship between continuous quantity and number must be explicated. That in turn requires defining the concept of a continuous quantity.

The quantitative variables treated within the physical sciences are, for the most part, accepted as being continuous. In short, this means that there are no *gaps* between any pairs of values within these variables. In physical theories, for example, space and time are thought of as continuous rather than as fundamentally granulated.

At first sight it might be thought that if a quantitative variable also satisfies condition (10) then it must be continuous.

- (10) If X and Y are any values of Q such that $X > Y$ then there exists Z in Q such that $X > Z > Y$ (density).

This condition means that between any two different values of a quantitative variable there lies a third. Such a variable is infinitely divisible (in principle) and so it might be thought to be continuous. However, it is not. The rational numbers are dense but they are not continuous.

Obviously, the natural numbers are not dense. Between any two consecutive natural numbers, n and $n+1$, there is no third value. However, the *rational* numbers are dense. The (positive) rational numbers are all the numbers n/m , where n and m are any natural numbers. They are, in short, the whole numbers together with all of their fractional parts. Given any two rational numbers, n/m and p/q , there will be the rational number $1/2(n/m + p/q) = (nq + mp)/2mq$ falling between them. Hence, they are dense.

If condition (10) captured the meaning of continuity then all lengths (which are presumed to be continuous) would be measurable by rational numbers. However, it is well known that they are not. If the length of the side of a square is one unit then the length of the diagonal is $+\sqrt{2}$ units, and $+\sqrt{2}$ is not a rational number (i.e., there are no natural numbers, n and m , such that $+\sqrt{2} = n/m$). So

relative to any unit length, X , there will be gaps between the lengths measurable by rational numbers.

If not in density, then wherein does continuity reside? Consider the lengths measurable by all rational numbers (relative to some unit). They will fall into two classes: those whose measure is less than $+\sqrt{2}$, and those whose measure is greater than $+\sqrt{2}$. The former class has no greatest member, for there is no greatest rational number less than $+\sqrt{2}$. As they increase in value the lengths within this class get closer and closer to being $+\sqrt{2}$ units but never reach it. Similarly, the latter class has no smallest member, for there is no smallest rational number greater than $+\sqrt{2}$. As they decrease in value the lengths within this class also get closer and closer to being $+\sqrt{2}$ units but never reach it. Between these two classes is the length of the diagonal of the unit square (the square whose sides are one unit long). This length is the smallest length greater than any in the first class, and the largest length smaller than any in the second class. It therefore fills the gap between these two classes. What is needed is some general way of specifying such gap-fillers. Of course, $+\sqrt{2}$ is not the only one.

These gap-fillers may be specified along the following lines. The class of all lengths whose measure is less than $+\sqrt{2}$ is *bounded above* by the length of the diagonal of the unit square. That is, all members of this class are less than that length. Furthermore, it is the *least upper bound* of that class because it is the smallest length greater than any in that class. It is through generalizing this concept of a least upper bound that the concept of continuity may be captured.

Let α be any nonempty set of lengths, then α is *bounded above* if there exists a length, Y , such that for all Z in α , $Z \leq Y$. Y is then an *upper bound* of α . A length, X , which is an upper bound of α is a *least upper bound* of α if it is such that for every upper bound of α , Y , $X \leq Y$. Then the length variable is continuous if and only if every nonempty set of lengths which has an upper bound has a least upper bound. If this condition is satisfied then there will be no gaps in the length variable like that into which the length of the diagonal of the unit square falls. Thus, a quantitative variable, Q , which is dense is also *continuous* if and only if,

- (11) Every non-empty set of values of Q which has an upper bound has a least upper bound.

CONTINUOUS QUANTITIES AND REAL NUMBERS

The view stemming from Campbell (1920) and formalized by Krantz et al. (1971), is that quantities are measurable because they are *isomorphic to a subsystem of the real numbers*. That is, they have a structure similar to that of a portion of the real number system. Krantz et al. prove that a variable satisfying conditions (1) – (9) has this relation to the real numbers. As they then go on to show, in the spirit of Campbell's theory, this fact enables numerical assignments to be made

to the values of the variable, and such assignments are similar to those used in measurement. True as these observations are, it is a mistake to think that they reveal the nature of measurement. For the connection between continuous quantities and real numbers is much more intimate than the representationalists suggest. Bostock (1979) shows that *ratios* of values of a continuous quantity are identical in structure to the real number system. Hence, he boldly argues, such ratios are the real numbers. If true, this means that in measurement numbers are not assigned to values of a quantitative variable, rather ratios of these values *are* numbers, and it is because of this fact that we find numbers in measurement. Measurement is seen to be the discovery of numerical relationships between the values of a quantitative variable. This position has the attractive advantage of showing that the numerical statements reporting measurements are thoroughly empirical, observational reports. This is in sharp contrast to the representational theory which implies that measurement statements, because they are in part numerical, are not direct reports of observations (cf. Adams, 1979). This position also links measurement exclusively to quantity and, so, outlaws the quite promiscuous concept of measurement hitherto encouraged within psychology.

It is convenient to begin with some brief comments about numbers themselves. There is a traditional, though no longer widely held, view according to which numbers are ratios of quantities (e.g., "the abstracted ratio of any quantity to another quantity of the same kind, which we take for unity" (Newton, 1728/1967 p. 2)). On this view, the natural, integral, and rational numbers all emerge as ratios of the quantity, aggregate magnitude. Any natural number, n , is the magnitude of an n -membered aggregate relative to that of a single-membered aggregate; any integer, $+$ or $-$, is the magnitude of a positive or negative difference of n members between aggregates relative to that of a single-membered aggregate; and any rational number, n/m , is the magnitude of an n -membered aggregate to that of an m -membered aggregate. The two great strengths of this view are that it makes the numbers empirical relations, and makes them all the same kind of relations, (ratios) (see Forrest & Armstrong, 1987, for a more philosophical treatment of this view). As will be shown the same kind of view can be held about the real numbers.

Modern set-theoretical accounts of the number systems have departed from this simplicity. Typically, the natural numbers are thought of as sets constructed upon the empty set, $\emptyset$ (i.e., $0 = \emptyset$ and $n = \{n-1\}$); the integers as sets of ordered pairs of natural numbers; and the rationals as sets of ordered pairs of integers. Of interest here is the way in which the work of the German mathematician, R. Dedekind, enabled the real numbers to be constructed as infinite sets of rational numbers, for this work also shows that they may also be taken to be ratios of quantities.

Dedekind noted that each real number cuts the rational numbers into two classes. For example, $+\sqrt{2}$ cuts the positive rational numbers into those whose square is less than two and those whose square is greater than two. Thus, he suggested, $+\sqrt{2}$ is just the cut between these two classes of rational numbers

(see Dedekind, 1909). Because the location of this cut is uniquely determined by just one of these classes (say, those whose square is less than two) $+\sqrt{2}$ may be identified with that class. More generally, all real numbers, both rational and irrational alike, may be thought of as cuts between classes of rational numbers, and may be identified with infinite classes of rational numbers.

Such cuts became known as "Dedekind cuts" and Bostock (1979) gave the following general definition of them:

A cut is in effect a set which contains *all* rational numbers less than some real number which is their least upper bound, and it may be defined (without mentioning real numbers) as a non-empty set of rationals such that (i) there are some rationals not in the set, (ii) if any rational is in the set then all rationals less than it are in the set, and (iii) if any rational is in the set then some rational greater than it is in the set. (p. 200)

Modern set-theoretical accounts of the real numbers identify them with Dedekind cuts. At first sight this definition of the real numbers seems far removed from the numbers we know through measurement. In fact, as will become evident, the connection is intimate.

Algebraically, the real number system is described as a *complete ordered field* (see, for example, Birkhoff & MacLane, 1965). Such a structure is characterized by the following 13 conditions. Let a , b , c , and d be any real numbers, then

$$1^*. \text{ If } a = b \text{ and } c = d \text{ then } a + c = b + d \text{ and } a \cdot c = b \cdot d;$$

$$2^*. \text{ } a + b = b + a \text{ and } a \cdot b = b \cdot a;$$

$$3^*. \text{ } (a + b) + c = a + (b + c) \text{ and } (a \cdot b) \cdot c = a \cdot (b \cdot c);$$

$$4^*. \text{ } a \cdot (b + c) = a \cdot b + a \cdot c;$$

$$5^*. \text{ } a + 0 = 0 + a = a;$$

$$6^*. \text{ } a \cdot 1 = 1 \cdot a = a;$$

$$7^*. \text{ } a + (-a) = (-a) + a = 0;$$

$$8^*. \text{ } a \cdot (1/a) = (1/a) \cdot a = 1;$$

$$9^*. \text{ One and only one of the following is true: } a = b \text{ or } a > b \text{ or } b > a;$$

$$10^*. \text{ If } a > b \text{ and } b > c \text{ then } a > c;$$

$$11^*. \text{ If } a > b \text{ then } a + c > b + c;$$

$$12^*. \text{ If } a > b \text{ and } 0 < c \text{ then } a \cdot c > b \cdot c;$$

$$13^*. \text{ Every nonempty set of real numbers that has an upper bound has a least upper bound.}$$

(Note that in these conditions $a + b$, $a \cdot b$, $-a$, $1/a$, 1, and 0 are also real numbers, $1 \neq 0$ and $=$ is taken to be reflexive, symmetric, and transitive).

This then is a brief sketch of the now widely accepted picture of the real numbers. As Bostock (1979) has argued it can be shown that ratios of values of

continuous quantitative variables possess all the properties the real numbers are said to have. That is, these ratios also form a complete ordered field.

To begin with, what is such a ratio? The answer may be illustrated by taking length as the example again. The concept may be explained by working from simple cases to the difficult ones. Let X and Y be any two lengths. What does it mean to say that the ratio of X to Y is n (i.e., that $X/Y = n$), where n is some natural number? Suppose that X is the length, six meters, and that Y is the length, three meters. What does $X/Y = 2$ mean? The fact that $X/Y = 2$ may be expressed as $X = 2Y$ (i.e., X is twice Y), which means that in this case the ratio of X to Y is just the fact that X is $2Y$. More generally, if $X/Y = n$ then this is the fact that $X = nY$ and because the concept of being nY has already been unfolded (see section on "Quantity," earlier in this chapter) this account is quite explicit. The relation in which X stands to Y here is that of one thing equalling n of another.

Very many ratios do not equal natural numbers. If X is the length, 6 meters, and Y , the length 9 meters, then $X/Y = 2/3$. What then does it mean to say that $X/Y = n/m$ (where n/m is some rational number)? The answer is, it simply means that $mX = nY$. In this case the relation in which X stands to Y is that of m of one thing equalling n of another.

These cases are simple compared with the next. What does X/Y mean when its value is irrational? Let X be the length of the diagonal of a square and Y the length of each of the sides. Then, $X/Y = +\sqrt{2}$. What does this mean? Let $R(<X/Y)$ be the class of all positive rational numbers less than X/Y . That is, $R(<X/Y)$ is the class of all positive n/m such that $n/m < X/Y$ (i.e., $nY < mX$). Of course, it is quickly recognized that $R(<X/Y)$ is just the Dedekind cut which $+\sqrt{2}$ is conventionally taken to be. Corresponding to each n/m in $R(<X/Y)$ is a true inequality, $nY < mX$. The meaning of $X/Y = +\sqrt{2}$ is just this infinite class of inequalities. That is, $X/Y = +\sqrt{2}$ means

$$Y < X$$

$$14Y < 10X$$

$$141Y < 100X$$

$$1414Y < 1000X$$

$$14142Y < 10000X$$

$$nY < mX \text{ (where } + (n/m)^2 < 2).$$

Thus, the relation between X and Y in this case, which is their ratio, is an infinite class of relations of the kind, m of one thing being greater than n of another. This reveals the intimate connection between Dedekind cuts and ratios of values of

continuous quantities. To state the matter in its most general form, let $R(>X/Y)$ be the class of all positive rational numbers greater than X/Y , and $R(=X/Y)$ be the class of all positive rational numbers equal to X/Y . Of course, $R(=X/Y)$ is empty if X/Y is irrational. As was recognized by Euclid, any two ratios of values of a quantitative variable, X/Y and W/Z , are identical (i.e., the same ratio) if and only if,

$$R(<X/Y) = R(<W/Z),$$

$$R(=X/Y) = R(=W/Z) \text{ and}$$

$$R(>X/Y) = R(>W/Z).$$

Then the ratio of X to Y is the infinite class of true relationships between X and Y of the following three kinds:

$$(i) \quad mX < nY;$$

$$(ii) \quad mX = nY; \text{ and}$$

$$(iii) \quad mX > nY$$

(for all natural numbers n and m). In cases where X/Y is irrational there will be no relations of kind (ii). But, if X/Y is not irrational then relations of kind (ii) will exist. Any one such relation entails all of the others, not only those of kind (ii), but those of kinds (i) and (iii) as well. In such cases X/Y may be identified with any relation of kind (ii) which is true of it. In all other cases (i.e., the irrational cases), X/Y means nothing more nor less than the two infinite classes of relations of kinds (i) and (iii) which are true of it. For convenience it may be identified with just one of these classes.

It will now be obvious from the previous discussion, that corresponding to each ratio of values of a continuous variable there is a Dedekind cut, $R(<X/Y)$. If the continuous variable is unbounded in at least one of two senses then corresponding to each Dedekind cut there is a ratio of values of that variable. These two senses are as follows:

(12a) Let Q be a continuous quantitative variable, then Q is *unbounded above* if for every X in Q and every natural number, n , nX exists.

(12b) Let Q be a continuous quantitative variable, then Q is *unbounded below* if for every X in Q and natural number, n , Y exists such that $nY = X$.

Further insight into the connection between these two systems (the ratios of values of any unbounded continuous quantity, Q , and the real numbers) can be gained by seeing how each of the basic concepts within the real number system,

(i) unity or 1;

(ii) zero or 0;

(iii) the addition relation: $a + b = c$;

- (iv) the multiplication relation: $a \cdot b = c$;
- (v) the additive inverse: $-a$; and
- (vi) the multiplicative inverse: $1/a$;

has a simple analogue within the system of ratios.

- (i) *Unity*: for any value X of Q , $X/X = 1$.

Specification of the zero element requires the concept of a *difference* between two values of Q . The difference between any two values X and Y of Q is the value, Z , such that either $X = Y + Z$ or $Y = X + Z$. If $X = Y + Z$ then the difference between X and Y is called " $+Z$ "; if $Y = X + Z$ then the difference between X and Y is called " $-Z$ ". Relative to the difference between X and Y the " $+$ " or " $-$ " preceding " Z " simply indicates whether it is the composition of Z with the second term (Y) or the first term (X) which equals the other term. If $X = Y$ then there is no such Z (i.e., there is no difference between them) and this is written as " 0 ". That is,

- (a) $X - Y = +Z$ if and only if $X = Y + Z$;
- (b) $X - Y = -Z$ if and only if $Y - Z = +X$; and
- (c) $X - Y = 0$ if and only if $X = Y$.

Then the ratio of any difference, $X - Y$, to any other value, W , is as follows:

- (d) $(X - Y)/W = + (Z/W)$ if and only if $X - Y = +Z$;
- (e) $(X - Y)/W = - (Z/W)$ if and only if $X - Y = -Z$; and
- (f) $(X - Y)/W = 0$ if and only if $X - Y = 0$.

Then,

- (ii) *Zero*: for any values, X and Y , of Q , $(X - X)/Y = 0$.
- (iii) *Addition*: for any values, U , V , W , X , Y , and Z , of Q $U/V + W/X = Y/Z$ if and only if for some value, T , of Q , $W/X = T/V$ and $Y/Z = (U + T)/V$. (For then, of course, $U/V + W/X = U/V + T/V = (U + T)/V = Y/Z$).
- (iv) *Multiplication*: for any values, U , V , W , X , Y , and Z , of Q $(U/V) \cdot (W/X) = Y/Z$ if and only if for some value T , of Q , $W/X = V/T$ and $Y/Z = U/T$. (For then, of course, $(U/V) \cdot (W/X) = (U/V) \cdot (V/T) = U/T = Y/Z$).
- (v) *Additive Inverse*: as implied by (b) and (e) above, for any values X and Y of Q , $-(X/Y) = -X/Y$.
- (vi) *Multiplicative Inverse*: for all values, X and Y , of Q $1/(X/Y) = Y/X$.

Given these correspondences it is a simple matter to prove that for any unbounded continuous quantity, the system of ratios of its values possesses precisely the structure of the real number system (for more details of this argument see Bostock, 1979). That is, this system of ratios is also a complete ordered field.

Although this does not prove that this system of ratios is the real number system, it means that there is no formal impediment in the way of such an identification. All one then needs are positive arguments for making it. Bostock (1979) argues that this identification makes it easy to explain why real numbers are applicable to such ratios. It also helps to explain why the theory of real numbers developed in physics close association with the theory of continuous quantities as it emerged in physics (for an account of this history see Kitcher, 1983).

It is widely believed by modern philosophers that numbers belong not to the spatio-temporal realm that we inhabit, but that they are "abstract entities", existing in some mysterious way outside of space and time. To an empiricist this is not a plausible view. Arithmetic is not simply a branch of pure mathematics. It is a theory whose application is more extensive and more successful than virtually any other theory. How would these truths of arithmetic be known or applied, if they were not truths about the same empirical realm that we inhabit and apply them to? As psychologists we believe that knowing requires a causal process between the fact known and the knower. Causal relations outside of space and time cannot be conceived of, for causal relations are partly spatio-temporal relations. Hence, from the fact that we know the truths of arithmetic, it follows that they are truths about empirical things.

What empirical things could they be truths about? If some truth, like a law of nature for example, is successfully applied to some situation, then it must be a truth about that situation. That is, it must be about some of the properties and relations found in that situation. Truths about real numbers are applied to situations where continuous quantities are believed to exist. Thus they may well be truths about such quantities. The fact that ratios of values of continuous quantities have the same structure as the real numbers, leaves them as the most compelling candidate for what the real numbers are. Such ratios it is concluded, instantiate real numbers, and the theory of real numbers is just the theory of such ratios at a level of abstraction that ignores other features of them (such as the fact that one of these continuous quantities is length, another mass, and so on).

It follows from this identification that claims about, say, the ratio of length X to length Y being some real number, r , are empirical claims and, if true, literally true. That is, reference to numbers in quantitative science is literal and not merely metaphorical, as the representational theory of measurement would have it.

MEASUREMENT

Having clarified these preliminary issues the meaning of measurement becomes obvious. Quite simply, measurement is a procedure for identifying values of quantitative variables through their numerical relationships to other values. Take

a simple example. We wish to know the length of a timber beam. This may be done by relating its length to that called a meter. It is found to be r meters long (where r is some real number). Here r is the ratio of the length of the beam to that of a meter and this *fact* enables the length of the beam to be characterized.

More generally, in measurement some (unknown) value of a quantitative variable is identified as being r units. A *unit of measurement* is simply a particular value of the relevant variable. It is singled out as that value relative to which all others are to be compared. Let the unit be Y and let the value to be measured be X . Then a measurement has the form $X = rY$. This, of course, is equivalent to $X/Y = r$. As has already been argued, statements of this form are empirical claims, each ratio of values of a continuous quantity being a real number. Measurement, then, is just the recognition of such facts.

If the value of Y is already known and the aim is to identify X , then this is achieved by noting that $X = rY$, for this fact characterizes X in terms of Y . Of course, Y itself must be known independently of measurement (otherwise an infinite regress ensues) and so, generally, Y is known by ostension. That is, the unit of measurement is instantiated in some object (called the *standard of measurement*) where it may be known by direct observation. If unknown values are then characterized by their numerical relations to a common unit, numerical relations between them may be inferred. For example if $X = rY$, and $Z = sY$, then $X/Z = r/s$. Measures relative to the same unit form a *scale of measurement*, and measurements on the same scale allow for further numerical relations to be deduced.

If r is an irrational number then its precise numerical specification requires an infinite number of terms (inequalities). Because this is not possible, measurement utilizes not the irrational real numbers themselves, but rational approximations to them. Every real number can be approximated to any level of accuracy (at least in principle) by some rational number. That is, if r is any real number and $e > 0$ the level of accuracy required, then there exists a rational number, n/m , such that $r + e/2 \geq n/m \geq r - e/2$. In measurement such rational approximation must be settled for: absolute accuracy is a myth.

This is a limitation forced on measurement simply because of its finite nature. The value of e is determined by the limitations of our procedures. All measurement procedures have finite resolution: there is a level of accuracy beyond which they cannot take us. One of the ways that science advances however, is by improving the accuracy of its measurements through the development of new techniques.

Measurement requires the development of procedures whereby values X and Y may be brought into comparison and their ratio assessed. Such procedures are the methods of measurement. The development of such methods is called *quantification*. Logically prior to quantification, though in practice the two steps may be combined, is the process of showing that the variable involved is quantitative. Unless this step is completed any measurement methods proposed for a particular variable cannot be accepted. The claim that such procedures measure this variable remains speculative. This is because the measurement of any variable

presupposes that it is a quantitative variable, and in the absence of confirming evidence this presupposition is at best speculation. This point is of particular importance when considering attempts at psychological measurement.

One does not need to show that all of conditions (1)–(13) are true. Confirmation of conditions (1)–(9) is sufficient. If a variable satisfies (1)–(9) but not (10)–(13) then ratios of its values may not constitute the full set of real numbers. However, ratios of its values will constitute a subset of the real numbers and hence its values will be measurable. Thus, the first step in attempts at psychological measurement is to test the hypothesis that the relevant variable (or variables) satisfy conditions (1)–(9).

It is surrounding this issue that much debate and confusion has developed within psychology over the past 50 years. It is not necessary to complete this step via the Campbellian process of fundamental measurement. To think so is to confuse the character of a quantity with the behavior of the objects manifesting its values. As will be shown in the next chapter, there is more than one way to confirm the existence of quantity.

This chapter has shown that to believe that a variable is quantitative is to believe that its values possess a definite kind of structure, that they are ordered and additive. Furthermore, it has been argued that if a quantitative variable is also both continuous and unbounded (either above or below) then ratios of its values constitute the system of real numbers. Hence, pairs of its values stand in numerical relations and this fact is what makes measurement possible and gives it its meaning. Therefore, the path to measurement must begin with the search for quantity.