

Searching For Quantity

To hypothesize that a particular variable is quantitative is one thing. To produce evidence supporting that hypothesis is another. The last chapter makes plain exactly what kind of structure one is hypothesizing when one takes a variable to be quantitative. The aim in this chapter is to examine some of the ways in which evidence corroborating that hypothesis may be obtained. Two very general ways are considered: extensive measurement and conjoint measurement.

Extensive measurement is the classical path to quantification. It was this path that led to the earliest known attempts at quantification. Roughly, it involves finding a way of combining objects that directly reflects the additive structure of the variable; for example, combining rods end to end in the same direction (for length), or combining marbles on a balance pan (for weight). So transparent is the evidence for quantity in this case that many have been tempted to identify this path to quantification with the desired end itself. This was Campbell's fallacy, though he was not the first nor the last to commit it. Attempts to apply the theory of extensive measurement to psychology have proved singularly unsuccessful.

Conjoint measurement involves a situation in which two variables (A and X) are noninteractively related to a third (P). It is not required that any of the variables be already quantified, although it is necessary that the values of P be orderable, and that values of A and X be independently identifiable (at least at a classificatory level). Then, via the order on P , ordinal and additive relations on A , X , and P may be derived. This path to quantification was first described within the psychological literature by Luce and Tukey (1964) and since then has been applied to a number of psychological variables. However, these applications have been relatively sparse and their success modest. It appears that the majority of psychologists do not realize the significance of this development. It is the aim of

this chapter to describe it in some detail and recommendations are made for a more rigorous application of conjoint measurement theory. Following this chapter, the remainder of the book is devoted to illustrating some uses of this theory within psychology.

EXTENSIVE MEASUREMENT

Extensive measurement depends on the discovery of ordering and concatenation relations on the objects that directly reflect the quantitative structure of the variable involved. A simple example of extensive measurement has already been described in chapter 2 in connection with weight measurement using a beam balance. There, indeed, the five propositions describing the behavior of a set of objects directly mirror conditions (1)–(9) given in chapter 3.

In this case, placing two objects a and b together on the same balance pan constitutes addition of their weights. That is, if the weight of a is X and b is Y , then the weight of the concatenated objects, $a \circ b$, is taken to be $X + Y$. Hence, the additive properties of the weight variable (conditions (4)–(9) of chapter 3) may be tested more or less directly. Conditions (7) and (9), because of their existential nature, may only be tested indirectly.

It is this approach to measurement that forms the basis of many of the measurement scales used in the physical sciences. As well as length and weight, it is applicable to the measurement of such variables as area, volume, duration (time), electrical resistance, plane angle, and many others. However, despite its importance in the physical sciences, it has proved to be of little use in psychology. One of the few attempts to apply it is that of Pollatsek and Tversky (1970) in their speculations surrounding the measurement of risk. In general, within psychology, the search for operations directly reflecting quantitative additivity has been in vain. Hence, no more will be said on this subject here. The reader is referred to Campbell (1920) and Ellis (1966) for more details on extensive measurement. Later in this chapter it is shown that extensive measurement is simply a special case of conjoint measurement.

CONJOINT MEASUREMENT

Luce and Tukey (1964) proposed the theory of conjoint measurement specifically for quantification within the social sciences. As will be explained, it provides a way for the identification of quantitative structure other than via operations of concatenation (or physical addition). In fact it enables quantitative structure to be detected via ordinal relations upon a variable. Although psychology lacks concatenation operations, it possesses ordinal relations aplenty.

The theory is about the kind of situation where a quantitative variable, P , is a noninteractive function of two other variables, A and X . For the moment

“noninteractive” may be understood to mean “additive” or “multiplicative,” although it is actually more general than this and a more accurate definition is given later. That is, conjoint measurement theory relates to situations of the kind $P = A + X$, or $P = A \times X$. Its application is specifically for those instances where none of P , A or X is already quantified. It requires that:

- (i) variable P possesses an infinite number of values;
- (ii) $P = f(A, X)$ (where f is some mathematical function);
- (iii) there is a simple order, $\geq$, upon the values of P ; and
- (iv) values of A and X can be identified (i.e., that objects may be classified according to the value of A and X they possess).

Let us call a system satisfying (i)–(iv) a *conjoint system*. Then, if $\geq$ on P satisfies *three special conditions* it follows that:

- (v) P , A and X are quantitative; and
- (vi) f is a noninteractive function.

The three special conditions are:

- (1) *Double Cancellation*;
- (2) *Solvability*; and
- (3) *the Archimedean condition*;

and these are described in detail below.

Before describing these conditions it may help to consider some examples of situations where (i)–(iv) are satisfied (or, believed to be so). Take a physical example, first. Let P be mass, A , density, and X , volume. Values of P may be ordered using, for example, a simple beam balance. Volumes, of course, could be classified by considering water displacement in some standard jar. And classifying objects according to the kind of material they are composed of classifies them according to density.

Or, to take a psychological example, suppose that P is performance on some task (say, the time it takes to run a maze), A is motivation and X is amount of prior practice. Of course, it would be a simple matter to order the performances and classify subjects according to motivation (e.g., duration of food or water deprivation) and number of previous practice trials.

Such conjoint systems are easily contemplated visually if they are thought of as constituting a matrix where the rows are values of A , the columns, values of X , and the cells, values of P . Let $a, b, c, \dots$ etc. be values of A , $x, y, z, \dots$ etc. be values of X and, since $P = f(A, X)$, the pairs, $ax, ay, \dots, cy, cz, \dots$ denote (possibly identical) values of P . Such a matrix is schematically represented

single line arrows representing the antecedent orders, and the double line arrow the consequent order). Within each set of six, if one of the instances of double cancellation is true, all of them are. However, between sets the instances of double cancellation are logically independent of one another. Thus, within any 3×3 matrix there are six independent tests of the double cancellation condition, this condition being false if in any of the diagrams shown in Fig. 4.3, the antecedent order relations hold while the consequent one does not; otherwise it is satisfied. Obviously, satisfaction of double cancellation (in a conjoint matrix, even a 3×3 one) is no trivial matter.

As will be shown later, there is an even more general notion of double cancellation imposing even stronger constraints on the order within a conjoint matrix. However, double cancellation, as stated here is sufficient for the present purpose.

SOLVABILITY

The solvability condition requires that the variables A and X be complex enough to produce any required value of P . It is formally stated as follows (cf. Krantz et al., 1971).

The order $\geq$ on P satisfies *solvability* if and only if (i) for any a and b in A and x in X , a value of X exists (call it y) such that $ax = by$ (i.e., both $ax \geq by$ and $by \geq ax$); and (ii) for any x and y in X and a in A , a value of A exists (call it b) such that $ax = by$.

In other words, given any three of a , b , x , and y , the fourth exists so that the equation

$$ax = by$$

is solved. Hence, the name *solvability*.

Once again, the meaning of this condition can be grasped by considering the conjoint matrix. It is equivalent to the requirement that every value of variable P occur in each row and each column of the matrix.

Thinking in terms of the relationship $P = A + X$, solvability implies that either values of A and X are equally spaced (as are the natural numbers) or else they are dense (as are the rational numbers).

THE ARCHIMEDEAN CONDITION

Remember that the Archimedean condition, as stated in chapter 3, ensures that no value of a quantitative variable is infinitely larger than any other value. Its meaning here is essentially the same, although in this context its expression is slightly more complex. Thinking in terms of $P = A + X$ again, a general idea of its content may be stated as follows. Conjoint measurement enables quantification of *differences* between values of A , between values of X and between values of

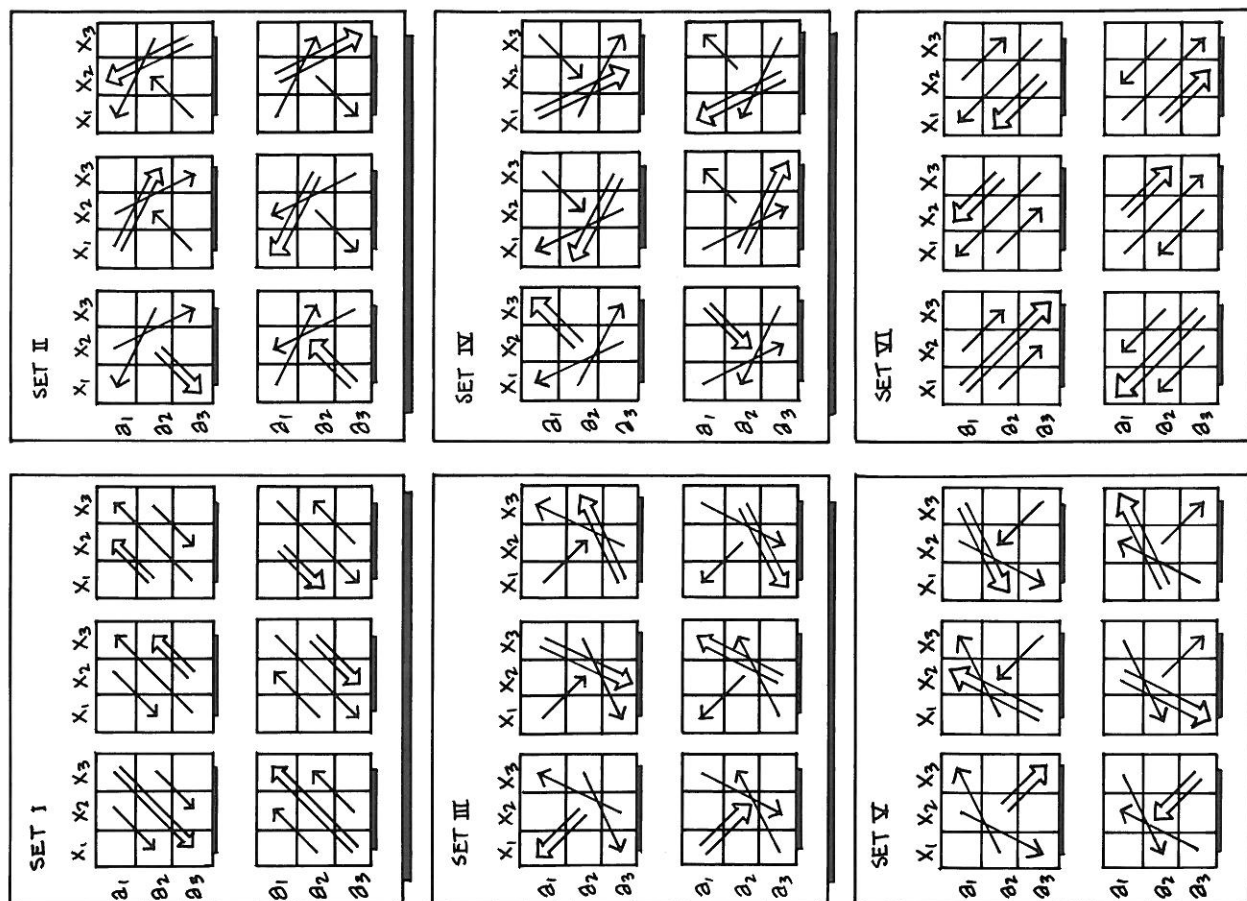


FIG. 4.3 The 36 different substitution instances of double cancellation in a 3×3 matrix. Single line arrows between cells represent the antecedent order relations ($\geq$) and double line arrows the consequent order relations. Double cancellation means that if the orders indicated by the single line arrows hold then so must those indicated by the double line arrows.

P. Limiting attention to *A*, the Archimedean condition means that no difference between any two values of *A* is infinitely larger than the difference between any other two values of *A*.

In order to introduce the concept of a difference between values of *A* and between values of *X*, attention must be drawn to the fact that if the simple order, $\geq$, on *P* satisfies both double cancellation and solvability, then it entails an ordering on *A* and *X*. Let *a* and *b* be any values of *A*, and *x* be any value of *X* such that $ax \geq bx$ (this is illustrated in Fig. 4.4). Let *z* be any other value of *X*. Now, if $\geq$ satisfies solvability then there exists a value, *c*, of *A* such that $bz = cx$.

Also, if $\geq$ satisfies double cancellation then both

$$ax \geq bx$$

$$\text{and } bz = cx$$

entail that $az \geq cx$.

(This is an instance of double cancellation that differs from the statement given earlier only in that $y=x$). Because $\geq$ on *P* is transitive (this is part of its being a simple order), it follows from the fact that,

$$az \geq cx$$

$$\text{and } cx = bz$$

that $az \geq bz$.

What has been proved here is that for any *a* and *b* in *A*, and *x* in *X*, if $ax \geq bx$ then for any *z* in *X* $az \geq bz$.

This means that the order between *ax* and *bz* is the same as that between *ay* and *by*, and that between *az* and *bz*, and so on. That is, the order between an *a*-cell and a *b*-cell in the same column of the conjoint matrix, is the same for all columns. In other words it is *independent* of which column we are to inspect (see

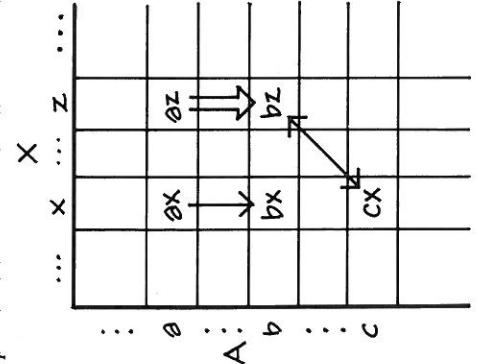


FIG. 4.4 Given that $\geq$ on *P* satisfies transitivity, double cancellation and solvability it follows that it also satisfies independence.

Fig. 4.5). In this sense then, any values, *a* and *b*, of *A* are ordered: $a \geq b$ if and only if for any (or all) *x* in *X*, $ax \geq bx$.

A similar chain of reasoning applies to any pair of values, *x* and *y*, of *X*: $x \geq y$ if and only if for any (or all) *a* in *A*, $ax \geq ay$.

This property of the order $\geq$ on *P* is called *independence* (cf. Krantz et al., 1971) and its importance is that not only does $\geq$ order values of *P*, it may be used to infer an order on the values of *A* and the values of *X*.

Given the fact that if a simple order on *P* satisfies double cancellation and solvability, then it also satisfies independence and that, therefore, the order of values of *A* and *X* is determined, the concept of the *difference* between values of *A* and between values of *X* may be explained. Let *a* and *b* be any two values of *A* such that $a > b$ (i.e., $a \geq b$ and not $b \geq a$), and let *x* and *y* be any two values of *X* such that $x > y$.

Then, of course, given that $\geq$ on *P* satisfies independence and transitivity it follows that

$$ax > ay$$

$$ay > by$$

$$ax > bx$$

$$bx > by$$

$$\text{and } ax > by$$

(see Fig. 4.6) However, what about the order between *ay* and *bz*? It is not determined by independence, so it may be true that either $ay \geq bz$ or $bz > ay$.

Suppose that $bz > ay$. Each of *bz* and *ay* differ from *ax* on only a single component (i.e., on just *A* or *X*). Thinking in terms of the relationship $P = A + X$ this is clearly seen. That is, if

$$bx = b + x,$$

$$ay = a + y,$$

$$\text{and } ax = a + x,$$

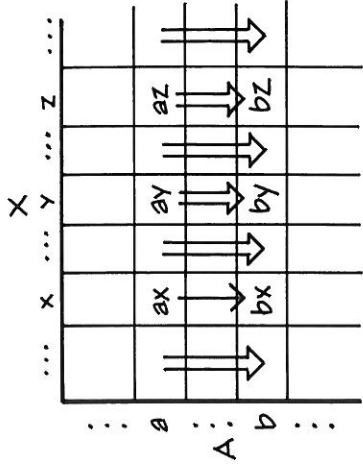


FIG. 4.5 The independence of $\geq$ on *A* from *X* means that for any *a*, *b* in *A* and *x* in *X*, if $ax \geq bx$ then for every other value of *X*, *y*, *z*, ..., $ay \geq by$, $az \geq bz$, ...

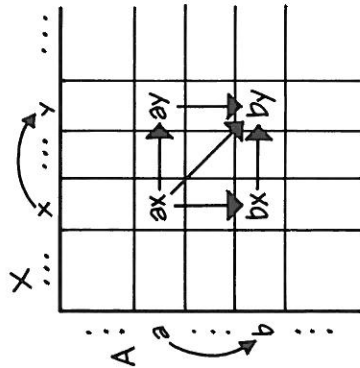


FIG. 4.6 Given independence and transitivity, if $a > b$ and $x > y$ then $ax > ay$, $ax > bx$, $ay > by$, $bx > by$ and $ax > by$.

then the difference between ax and bx is simply $(a + x) - (b + x) = a - b$ and the difference between ax and ay is simply $(a + x) - (a + y) = x - y$.

Now, if $bx > ay$ then the difference between ax and ay must exceed the difference between ax and bx (because $ax > bx > ay$). That is, $x - y > a - b$.

At first sight this claim may seem to be nonsense, for $x - y$ is a difference between values of X , $a - b$ is a difference between values of A , and different quantitative variables are not commensurate. It is akin to claiming that a weight difference exceeds a length difference. However, just as the ordering of values of A and X is dependent upon the order on P , so is the ordering of differences within A and X . That is, these orders are relative to their effects on P . The claim that $x - y > a - b$ means nothing more in this context other than that the effect on P of moving from x to y in X , exceeds the effect on P of moving from a to b in A . In conjoint measurement the variables A and X are quantified relative to their effects on P .

Considering any interval on, say, variable A , such as $a - b$, the solvability condition ensures that, relative to any value, x , of X , there exists another value of X , z , such that $a - b = x - z$. That is, $bx = az$.

Thus, relative to $a - b$ of A there will exist a series $x_1, x_2, \dots, x_n$ of values of X such that

$$a - b = x_1 - x_2 = x_2 - x_3 = \dots = x_{n-1} - x_n.$$

Such a series Krantz et al. (1971) refer to as a *standard sequence* and it is simply a sequence of equal steps along the ordered values of variable X . Similarly, relative to $x - y$ of X there will be a standard sequence of values of A : $a_1, a_2, \dots, a_m$. Now the Archimedean condition can be stated with some precision. If $a - b$ and $c - d$ are two differences between values of A such that

$$a - b < c - d$$

then there exists a natural number n such that

$$n(a - b) \geq c - d$$

(where $n(a - b)$ is the standard sequence $a_1, a_2, \dots, a_{n+1}$, where $a = a_1$, and $b = a_2$). Similarly, any difference between values of X , no matter how large, cannot be so great relative to any difference between other values of X , that some finite standard sequence based on the smaller difference does not exceed it. No difference within A or X is infinitely larger than any other difference within A or X .

CONJOINT NONINTERACTIVE QUANTITATIVE STRUCTURES

If the variables, A , X , and P form a conjoint system in which the simple order, $\geq$, on P satisfies (1) double cancellation, (2) solvability, and (3) the Archimedean condition, then A , X , and P are quantitative variables and P is a noninteractive function of A and X .

A proof of this claim is given by Krantz et al. (1971). Essentially, the proof involves showing that certain pairs of values of A and X (what might be called "differences" between their values) stand in the ordinal and additive relations necessary for quantity. Likewise, the same may be shown for pairs of values of P . Then it is shown that P is a noninteractive function of A and X .

This concept of a noninteractive function requires some explanation. The function f such that $P = f(A, X)$ is noninteractive if and only if there exist monotonic functions f_1, f_2 and f_3 such that

$$\begin{aligned} f_1(P) &= f_2(f(A, X)) \\ &= f_2(A) + f_3(X). \end{aligned}$$

That is, f is noninteractive if and only if under some monotonic transformations it reduces to an additive function of A and X (or monotone transformations of them). For example, if

$$\begin{aligned} P &= f(A, X) = A \times X \\ \text{then } \log P &= \log(A \times X) \\ &= \log A + \log X \end{aligned}$$

and because the logarithmic transformation is monotonic, the function f is noninteractive. Of course, the functions

$$\begin{aligned} P &= A + X \text{ and} \\ P &= A \times X \end{aligned}$$

are the simplest of noninteractive functions, and the entire class of such functions is infinitely large. Unless there are special reasons for not doing so, the noninteractive

relationship between P , A , and X may always be expressed in the simple additive or multiplicative forms. Whichever is chosen is merely a matter of convenience.

This may appear to be a strange recommendation. Surely the question of how the variables A and X relate to P is a *factual* matter rather than one of convenience? It is, of course, a factual matter, but the same variable may manifest quite different additive relations. This point is seen more clearly if a variable is distinguished from the quantitative variable of which it is a part. A variable, as already noted, is a class of mutually exclusive properties or relations. A quantitative variable is a variable together with ordinal and additive relations satisfying conditions (1)-(9) for quantity. Then a variable, V , might form part of two quite different quantitative variables, $Q_1 = (V, \geq, +)$ and $Q_2 = (V, \geq, *)$ (where $+$ and $*$ are distinct ternary relations on V satisfying conditions (4)-(9) for quantity). Indeed, if a variable forms part of *one* quantitative variable, then it forms part of infinitely many.

For example, let $Q_1 = (V, \geq, +)$ be a continuous quantitative variable and let U be any value of V . Then there will exist the relation $*$ such that for any X , Y , and Z in V

$$X * Y = Z \text{ if and only if } (X/U + 1) \cdot (Y/U + 1) = (Z/U + 1).$$

It can easily be shown that $Q_2 = (V, \geq, *)$ is also a quantitative variable. Of course, $*$ is not the only other ternary relation on V satisfying conditions (4)-(9) for quantity.

Thus, if P , A , and X satisfy the conditions for conjoint measurement then P is a noninteractive function of A and X , but which noninteractive function is chosen from the infinitely many present is a matter of convenience. It depends on what additive relations within A , X , and P are identified. It is usually most convenient to choose them such that $P = A + X$, but this is not always the case.

For example, let P be mass, A density, and X volume. In this case the conjoint system satisfies double density cancellation, solvability, and the Archimedean condition and there is a weak order on P . Because it is already accepted within physics that mass = density $\times$ volume, that would be the most appropriate way to express the relationship between these variables. However, it could be expressed as mass* = density* + volume*, in which case mass and mass*, for example, are the same variable but involve quite different additive relations between values of the variable (i.e., they are different quantitative variables).

QUANTIFICATION BY CONJOINT MEASUREMENT

Quantification by conjoint measurement involves the development of procedures for testing the hypothesis that a given conjoint system is a noninteractive, quantitative structure. Although it may be difficult to test double cancellation, especially in large matrices (remember that there are six independent tests of double cancellation in each 3×3 matrix and the number of 3×3 matrices in even a small conjoint

matrix is large; e.g., in a 5×5 matrix there are 100 different 3×3 matrices) this condition is *directly* testable. This is not the case with solvability and the Archimedean condition.

Consider solvability. Suppose that relative to some a and b in A and x in X no y can be found such that $ay = bx$. This does not mean that there is none, for limitations of time and technology may make it difficult to locate some values of a variable. Thus solvability is not directly falsifiable.

Similarly, for a given difference $x - y$ the standard sequence $a_1, a_2, \dots, a_i$ may not be able to be shown to be finite. However this again could be due to limitations of time and technology, so the Archimedean condition as well is not directly testable.

This does not mean that these conditions are untestable or "unempirical." They may be tested indirectly, as will be shown. Scott (1964) has shown that any finite subsystem of a noninteractive conjoint structure must satisfy a finite set of generalized cancellation conditions. These, like double cancellation, are directly testable (at least in principle). So if they prove true for any finite set of data (i.e., for any finite conjoint measurement matrix) then this must be indirect evidence that solvability and the Archimedean condition are true. The meaning of this claim will become apparent once the concept of generalized cancellation conditions is explained.

Thus far the concept of double cancellation has been explained. In an earlier section a condition called *independence* was introduced. It may be regarded as *single cancellation*. As well as single cancellation and double cancellation there is triple cancellation. A version of this conditions states that if a, b, c , and d are any values of A and w, x, y , and z are any values of X then $\cong$ on P satisfies *triple cancellation* only if

$$az \cong by$$

$$\text{and } cy \cong dz$$

$$\text{and } dw \cong cx$$

$$\text{then } aw \cong bx.$$

An instance of this condition is illustrated in Fig. 4.7.

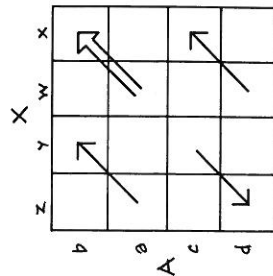


FIG. 4.7 An instance of triple cancellation.

Beyond triple cancellation there is quadruple, or fourth-order, cancellation, and so on. Generally, there is n th-order cancellation for any value of n . If these cancellation conditions are to be tested then there must be a general statement of n th-order cancellation. Such a condition is complex and not easily stated.

Let $a_1, a_2, \dots, a_p, \dots, a_{n+1}$ be any $(n+1)$ termed sequence of values of A and $x_1, x_2, \dots, x_p, \dots, x_{n+1}$ be any $(n+1)$ termed sequence of values of X . A degenerate instance (i.e., a trivially true instance) of n th-order cancellation may be expressed as follows:

$$\begin{array}{ll} \text{if} & 1. \quad a_1 x_1 \geq a_2 x_2 \quad \text{and} \\ & \cdot \\ & \cdot \\ & \cdot \\ & i. \quad a_{i+1} x_{i+1} \geq a_{i+1} x_{i+1} \quad \text{and} \quad (i = 2, \dots, n-1) \\ & \cdot \\ & \cdot \\ & \cdot \\ & n. \quad a_{n+1} x_{n+1} \geq a_{n+1} x_{n+1} \quad \text{then} \\ & \quad a_1 x_1 \geq a_2 x_2. \end{array}$$

This condition is trivially true because its logical form is if p and q_1 and $\dots$ and q_n then p (which is a tautology) and, incidentally, each of $q_1, \dots, q_n$ is also a tautology. However, it may be used to provide the basis for a quite general statement of n th-order cancellation. The terms within the antecedent of the degenerate condition may be set out in an $n \times 4$ matrix as follows:

$$\text{Matrix 1} = \begin{bmatrix} a_1 & x_1 & a_2 & x_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{i+1} & x_{i+1} & a_{i+1} & x_{i+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1} & x_{n+1} & a_{n+1} & x_{n+1} \end{bmatrix}$$

If the terms within any column of Matrix 1 are permuted then, taking the terms within each row as the terms of an inequality of the form $a_i x_{ii} \geq a_{ii} x_{iv}$ (where a_i is the term in the first column, x_{ii} the term in the second, a_{ii} the term in the third and x_{iv} the term in the fourth), a different set of n inequalities will be generated. For example, let the order of the terms in column four of Matrix 1 be reversed. Then Matrix 1 becomes:

$$\text{Matrix 2} = \begin{bmatrix} a_1 & x_1 & a_2 & x_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{i+1} & x_{i+1} & a_{i+1} & x_{n-i+2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1} & x_{n+1} & a_{n+1} & x_2 \end{bmatrix}$$

The corresponding set of n inequalities is then,

$$\begin{array}{ll} 1. & a_1 x_1 \geq a_2 x_{n-1} \\ \cdot & \\ \cdot & \\ \cdot & \\ i. & a_{i+1} x_{i+1} \geq a_{i+1} x_{n-i+2} \quad (i = 2, \dots, n-1) \\ \cdot & \\ \cdot & \\ n. & a_{n+1} x_{n+1} \geq a_{n+1} x_2 \end{array}$$

If $\geq$ on P is a noninteractive quantitative, conjoint structure then if each of these n inequalities is true it follows that $a_1 x_1 \geq a_2 x_2$. This argument is nontrivial, for the conclusion is not identical to any of the premises. Similarly, any other permutation of the terms in the columns of Matrix 1 produces a new matrix and, hence, a new set of n inequalities. Each of these sets of n inequalities in turn also implies (given that $\geq$ on P is a noninteractive, quantitative, conjoint structure) $a_1 x_1 \geq a_2 x_2$ and providing the intra-column permutations are not all identical, the argument so produced is nontrivial.

It is easy to see that in each case $a_1 x_1 \geq a_2 x_2$ must follow. As noted above, if $\geq$ on P is a noninteractive conjoint structure then $P = A + X$. That is, each value of P, ax , may be expressed as the sum of its row and column factors, that is, as $a + x$. Thus, if $ax \geq by$, then $a + x \geq b + y$.

Hence, the basic, degenerate n th-order cancellation condition may be expressed as,

$$\begin{array}{ll} \text{if} & 1. \quad a_1 + x_1 \geq a_2 + x_2 \quad \text{and} \\ & \cdot \\ & \cdot \\ & \cdot \\ & i. \quad a_{i+1} + x_{i+1} \geq a_{i+1} + x_{i+1} \quad \text{and} \\ & \cdot \\ & \cdot \\ & \cdot \\ & n. \quad a_{n+1} + x_{n+1} \geq a_{n+1} + x_{n+1} \quad \text{then} \\ & \quad a_1 + x_1 \geq a_2 + x_2. \end{array}$$

Now, as is well known, if

$$p \geq q \text{ and}$$

$$r \geq s \text{ then}$$

$$p + r \geq q + s.$$

Thus, the n antecedent inequalities may be summed, giving

$$\begin{aligned} a_1 + x_1 + \dots + a_{i+1} + x_{i+1} + \dots + a_{n+1} + x_{n+1} &\geq \\ a_2 + x_2 + \dots + a_{i+1} + x_{i+1} + \dots + a_{n+1} + x_{n+1} & \end{aligned}$$

(call this the summed inequality) and if terms common to both sides are cancelled the result is $a_1 + x_1 \geq a_2 + x_2$.

When a new set of n inequalities is generated by permuting terms within the columns of Matrix 1 all that is altered is the inequality a given term occurs within. The terms appearing on the left-hand side and the right-hand side of the summed inequality must remain constant. Hence, the same terms will always cancel, leaving

$$a_i + x_i \geq a_2 + x_2$$

$$\text{i.e., } a_i x_i \geq a_2 x_2.$$

So corresponding to each distinct permutation of terms within the columns of Matrix 1 is an n th-order cancellation condition, that must be true if $\geq$ on P is a noninteractive quantitative conjoint structure. The full set of different n th-order cancellation conditions may be generated by permuting just the terms within columns 2, 3, and 4 of Matrix 1. Because the number of permutations within each column is $n!$, there will be $(n!)^3$ n th-order cancellation conditions for each different set of values $a_1, a_2, \dots, a_{n+1}$ and $x_1, x_2, \dots, x_{n+1}$ of A and X , that must all be true within any finite conjoint measurement matrix.

Within this set of cancellation conditions each lower order cancellation condition is subsumed within n th-order cancellation. That is, independence (or, single cancellation) is a special case of double cancellation, double cancellation of triple cancellation, and so on. In the case of double cancellation the degenerate schema is,

$$\text{if } a_i x_i \geq a_2 x_2 \text{ and}$$

$$a_j x_j \geq a_3 x_3 \text{ then}$$

$$a_i x_i \geq a_2 x_2.$$

For each different set of values a_1, a_2, a_3 of A and x_1, x_2, x_3 of X this gives rise to $(2!)^3 = 8$ distinct double cancellation conditions. One of these is the condition stated in the section on "Double Cancellation." This is obtained by permuting only the terms in columns two and three, that is,

$$\text{if } a_i x_i \geq a_3 x_2 \text{ and}$$

$$a_j x_j \geq a_2 x_3 \text{ then}$$

$$a_i x_i \geq a_2 x_2.$$

Another special case of double cancellation is obtained by permuting just the terms in columns three and four. This gives

$$\text{if } a_i x_i \geq a_3 x_3 \text{ and}$$

$$a_j x_j \geq a_2 x_2 \text{ then}$$

$$a_i x_i \geq a_2 x_2,$$

the transitivity of $\geq$ on P .

Still another version of double cancellation is obtained by permuting only the terms in columns two and four, that is,

$$\text{if } a_i x_i \geq a_2 x_3 \text{ and}$$

$$a_j x_j \geq a_3 x_2 \text{ then}$$

$$a_i x_i \geq a_2 x_2.$$

If $x_1 = x_2$ then this becomes,

$$\text{if } a_i x_i \geq a_2 x_3 \text{ and}$$

$$a_j x_j \geq a_3 x_i \text{ then}$$

$$a_i x_i \geq a_2 x_i$$

(note that only a_3 and x_3 are cancelled). This, of course, is logically equivalent to,

$$\text{if } a_i x_i \geq a_2 x_3 \text{ then}$$

$$a_i x_i \geq a_2 x_i$$

(for the second premise (" $a_i x_i \geq a_3 x_i$ ") is a tautology and, so, may be dropped) that is, independence, or single cancellation. Similarly, double cancellation is a special case of triple cancellation and so on.

In general, when A and/or X are infinite there are infinitely many different cancellation conditions. In practice, however, an application of the theory of conjoint measurement requires operating on finite subsets of values of A and X . The prediction is, if $\geq$ on P is a noninteractive conjoint structure then all n th-order cancellation conditions will be true. This prediction follows from the fact that the simple order, $\geq$, on P satisfies double cancellation, solvability and the Archimedean condition. Although the fact that $\geq$ is a simple order and satisfies double cancellation is equivalent to certain cancellation conditions, it alone does not entail all n th-order cancellation conditions. Solvability and the Archimedean condition are premises necessary for that conclusion. Hence, satisfaction of all

n th-order cancellation conditions in some finite conjoint measurement data matrix provides indirect support for the truth of these two conditions. Thus, they are testable.

The most compelling evidence for $\cong$ on P being a noninteractive, conjoint structure is obtained through the construction of standard sequences on both A and X . If an $n \times n$ matrix of values of P is constructed where $a_1, a_2, \dots, a_n$ and $x_1, x_2, \dots, x_n$ form standard sequences then it is a finite model of a noninteractive conjoint structure. Within that finite model double cancellation, solvability and the Archimedean condition are all obviously true and hence A , X , and P are readily seen to be quantitative. Furthermore, within such a structure the numerical relationships between values of A and between values of X are transparently obvious. One can see that, for example,

$$a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1}$$

and that

$$a_3 - a_1 = 2(a_2 - a_1), \text{ and}$$

$$a_4 - a_1 = 3(a_2 - a_1), \text{ and}$$

$$a_n - a_1 = (n-1)(a_2 - a_1).$$

Hence, it provides a clear basis for the measurement of A , X , and P . Without such a structure, A , X , and P cannot be measured with any accuracy. One is reduced to working with methods for solving sets of linear inequalities (e.g., the simplex algorithm) and these are necessarily less precise.

The construction of standard sequences on A and X is no trivial matter. It requires a degree of technological control over A and X that is often difficult to achieve in psychology. This will become evident in later chapters and it suggests that a major obstacle to psychological measurement is the technical control necessary in determining the values of variables.

THE GENERALITY OF CONJOINT MEASUREMENT

Conjoint measurement is a quite general approach to measurement. As will be indicated in the final chapter, it may be extended forwards beyond the three variable case (i.e., $P = f(A, X)$) to four or more variable cases (i.e., $P = f(A_1, \dots, A_n)$). In this section it will be shown how conjoint measurement may be extended backwards to cases involving one variable. As such it subsumes extensive measurement.

Extensive measurement is a special case of conjoint measurement. Let Q be the variable involved in extensive measurement. Then, within the conjoint measurement framework, $A = Q$, $X = Q$, and $P = Q + Q (= 2Q)$. That is, the values of A and X are the values of the variable Q itself and P are these values

as combined under the physical operation of addition (see Fig. 4.8). If the simple order on $Q + Q$ satisfies double cancellation, solvability and the Archimedean condition then Q is quantitative (i.e., satisfies conditions (1)-(9) for quantity). This means that satisfaction of the hierarchy of cancellation conditions within a finite substructure of $Q + Q$ supports the hypothesis that Q is quantitative. That is, in particular, it supports conditions (7) and (9), neither of which is directly testable.

However, the application of the theory of conjoint measurement to a single variable under some operation of physical addition is actually a more general notion than extensive measurement. There are variables that are not extensive in the ordinary sense and yet satisfy the conjoint measurement conditions given an appropriate concatenation procedure. One such is *velocity*.

Suppose that the velocity of object a relative to object b is x meters per second, and that the velocity of object b relative to d is y meters per second. What is the velocity of a relative to d ? According to Newtonian theory it is $x + y$ meters per second. Thus, within Newtonian theory velocity is extensively measurable. However, according to relativity theory the velocity of d relative to a is $(x + y)/(1 + xyc^{-2})$, where c is the velocity of light. So concatenating velocities in this way does not enable extensive measurement of velocity, for the velocity of a concatenation is not the sum of the concatenated velocities.

However, $(x + y)/(1 + xyc^{-2})$ (or $(x + y)/(1 + xy)$ if the unit of velocity is the speed of light) is a noninteractive function of x and y . That is, there exists a monotonic function, f , such that

$$f((x + y)/(1 + xy)) = f(x) + f(y).$$

Indeed, $f(x) = \tanh^{-1}(x) = 1/2 \log_e ((1+x)/(1-x))$ cf. Krantz et al. (1971). So, in fact, conjoint measurement reveals the additive structure of velocity even though velocity as ordinarily conceived of is not extensively measurable.

	...	x	...	y	...	z	...
:	:	:	:	:	:	:	:
x	...	x+x	...	x+y	...	x+z	...
:	:	:	:	:	:	:	:
Q y	...	y+x	...	y+y	...	y+z	...
:	:	:	:	:	:	:	:
z	...	z+x	...	z+y	...	z+z	...
:	:	:	:	:	:	:	:

FIG. 4.8 Extensive measurement is a special case of conjoint measurement: variable Q defines the row and column components and their sums (also variable Q) form the cell entries.

This kind of generalization of conjoint measurement to a single variable is of special interest to psychology. Although psychologists have not been able to locate suitable concatenation operations on which to base an application of the theory of conjoint measurement, they have been able to locate what they believe are *difference relations* within a single variable. That is, although structures of the kind $Q = Q + Q$ have not been found, it is believed that structures of the kind $Q = Q - Q$ have. Because difference is simply a special case of addition (i.e., since $Q - Q = Q + (-Q)$), it obviously follows that the theory of conjoint measurement can be applied to such structures to test the hypothesis that Q is a quantitative variable. This observation forms the basis for some of the applications of this theory in later chapters of this book.

CONCLUSION

Quantification in science is inseparable from the experimental method. The hypothesis that a particular variable is quantitative is a substantive hypothesis. It requires that the values of that variable manifest a definite kind of structure. With the possible exception of some quantitative variables open to extensive measurement, evidence for or against the quantitative hypothesis must be gained by experiment. The gathering of such evidence requires a high degree of experimental control and, often, sophisticated apparatus and methods of observation. It is no accident that the extension of quantification in physics from geometry and statics to dynamics, thermodynamics, and electrical phenomena went hand in hand with advances in experimental technique and apparatus. Psychological measurement, if it is to be, requires the same kind of advances.

In this chapter some of the paths to quantification have been considered. In particular, the method of conjoint measurement has been considered in some detail. Its application in psychology requires control over the values of the variables involved and methods for determining order. As will be shown in the following chapter, even these modest requirements present difficulties for psychology.

In the next three chapters, the application of conjoint measurement theory to some attempts at psychological measurement are considered. The logic of these applications is as follows. In this chapter, it has been shown that all variables satisfying the conditions for conjoint measurement are quantitative. What is shown in the following chapters is that each of the attempts at psychological measurement considered is based on a theory that predicts that the variables involved will satisfy the conditions for conjoint measurement. If those predictions are true then those variables are quantitative. This conclusion follows with the iron logic of the syllogism. So our major interest is in showing how, in each case, the prediction from the theory that the variables involved do satisfy the conditions for conjoint measurement, may be tested observationally. Also, in each case a small body of data will be considered in order to illustrate just how

such a test may be carried through. The aim is not so much to use this data to accept or reject the quantitative hypothesis. Such a decision remains a long way off. The aim is to acquaint the student with a way of proceeding when faced with the claim that this or that psychological variable is measurable.